Linear Algebra II

17/07/2009, Friday, 14:00-17:00

1 Gram-Schmidt process

Consider the vector space $C[-1,1]$, i.e. the vector space of continuous functions defined on the interval $[-1,1]$, and the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) By apply the Gram-Schmidt process, find an orthonormal basis for the subspace spanned by $\{1, x, x^2\}$.

(b) Find the coordinates of the function $1 + x^2$ in the orthonormal basis obtained above.

2 Diagonalization

(a) Diagonalize the matrix

$$M = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

(b) Is it possible to find an orthogonal matrix that diagonalizes M? Why?

3 Positive definite matrices

(a) Consider the function

$$f(x,y,z) = -\frac{1}{4}(x^{-4} + y^{-4} + z^{-4}) + yz - x - 2y - 2z.$$

(i) Verify that $(1,1,1)$ is a stationary point.

(ii) Determine whether this point is a local minimum, local maximum, or saddle point.

(b) Let A be a symmetric positive definite matrix and B be a symmetric nonsingular matrix. Show that

(i) A is nonsingular.

(ii) A^{-1} is positive definite.

(iii) $B^2 - 2I + B^{-2}$ is positive semi-definite.

(a) Let X and Y be two square matrices. Suppose that $XT = TY$ for a nonsingular matrix T. Show that the characteristic polynomials of X and Y are the same.

(b) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}.$$

Compute $A^{10000} + A^{9998}$.

5 Singular value decomposition

(a) Find the singular value decomposition of the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Consider the decomposition

$$\begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

(i) Is this a singular value decomposition? Why?

(ii) Find the closest (with respect to the Frobenius norm) matrices of rank 1 and 2.